COBOUNDING ODD CYCLE COLORINGS

DMITRY N. KOZLOV

Abstract. We give a very short self-contained combinatorial proof of the Babson-Kozlov conjecture, by presenting a cochain whose coboundary is the desired power of the characteristic class.

1. Preliminaries

The study of the following family of complexes has recently been undertaken in connection with equivariant obstructions to graph colorings.

Definition 1.1. *For any graphs T and G, $\mathtt{Hom}\,(T,G) \subseteq \prod_{x\in V(T)} \Delta^{V(G)}$ consists of all cells $\sigma = \prod_{x\in V(T)} \sigma_x$, such that for any $x,y \in V(T)$, if $(x,y) \in E(T)$, then (σ_x, σ_y) is a complete bipartite subgraph of G.*

In particular, the cells of $\mathtt{Hom}\,(T,G)$ are indexed by functions $\sigma : V(T) \to 2^{V(G)}$ satisfying that additional property, and $\dim \sigma = \sum_{v\in V(T)}(|\sigma(v)|-1)$. We refer the reader to the survey [4] for an introduction to the subject of $\mathtt{Hom}$ complexes.

The study of the complexes $X_{r,n} := \mathtt{Hom}\,(C_{2r+1}, K_n)$, $n \geq 3$, has been of special interest. Here for $r \in \mathbb{N}$, we let C_{2r+1} denote both the cyclic graph with $2r+1$ vertices and the additive cyclic group with $2r+1$ elements. The adjacent vertices of $v \in C_{2r+1}$ get labels $v+1$ and $v-1$. Taking the negative in the cyclic group gives an involution γ of the graph with a fixed vertex 0 and a flipped edge $(r, r+1)$. Then $(X_{r,n}, \gamma)$ is a $\mathbb{Z}_2$-space, hence the Stiefel-Whitney characteristic class $w_1(X_{r,n}) \in H^1(X_{r,n}/\mathbb{Z}_2; \mathbb{Z}_2)$ of the associated line bundle can be considered.

Theorem 1.2. (Babson-Kozlov conjecture). *We have $w_1^{n-2}(X_{r,n}) = 0$.*

The case $r = 1$ was settled in [2]. For $r \geq 2$, and odd n, it was proved in [3], see also [4], where the remaining case: $r \geq 2$, $n \geq 4$, n is even, was conjectured. The latter was then proved in [5, 6]. In the next section we give a short self-contained combinatorial proof of Theorem 1.2 covering all cases: we simply take a cochain representative of $w_1^{n-2}(X_{r,n})$ and certify that it is a coboundary.

First we fix notations. For $t \in \mathbb{N}$, we set $[t] := \{1, \dots, t\}$. For a cell complex X, we let X^d denote the set of d-dimensional cells of X. Since we are working over $\mathbb{Z}_2$, we may identify d-cochains with their support subsets of X^d. Then, the cochain addition is replaced by the symmetric difference of sets, denoted $\oplus$. For $S \subseteq X^d_{r,n}$ the coboundary operator translates to $\partial S = \oplus_{\sigma\in S}\{\tau \in X^{d+1}_{r,n} \,|\, \tau \supset \sigma\}$.

If r is even, set $t := r/2$, and $v_i := r - 2i + 1$, for $i \in [t]$, else set $t := (r+1)/2$, and $v_i := r + 2i - 1$, for $i \in [t]$. For any $v \in C_{2r+1}$, we set

$$A_v := \{\sigma \in X^{n-2}_{r,n} \,|\, \sigma(v) = [n-1]\},\ B_v := \{\sigma \in X^{n-3}_{r,n} \,|\, \sigma(v-1) \cup \sigma(v+1) = [n-1]\}.$$

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For $S \subseteq X^d_{r,n}$, set $q(S) := \oplus_{\sigma\in S}\{\mathbb{Z}_2\sigma\} \in C^d(X_{r,n}/\mathbb{Z}_2)$, where $\mathbb{Z}_2\sigma = \{\sigma, \gamma\sigma\}$. We see that $q(A_0) = \emptyset$, and $q(S \oplus T) = q(S) \oplus q(T)$, for any $S, T \subseteq X^d_{r,n}$. Furthermore, since $\tau \cap \gamma\tau = \emptyset$, for any $\tau \in X^{d+1}_{r,n}$, we have $q(\partial S) = \partial q(S)$, for any $S \subseteq X^d_{r,n}$.

It is easy to describe a cochain representing $w_1^{n-2}(X_{r,n})$. Let $\iota : K_2 \hookrightarrow C_{2r+1}$ be given by $\iota(1) = r$, $\iota(2) = r + 1$, where $V(K_2) = [2]$. This induces an algebra homomorphism $\varphi : H^*(\mathtt{Hom}\,(K_2, K_n)/\mathbb{Z}_2; \mathbb{Z}_2) \to H^*(X_{r,n}/\mathbb{Z}_2; \mathbb{Z}_2)$. It is well-known that $\mathtt{Hom}\,(K_2, K_n)/\mathbb{Z}_2 \cong \mathbb{RP}^{n-2}$. Let $\tau \in \mathtt{Hom}\,(K_2, K_n)^{n-2}$ be given by $\tau(1) = [n-1]$, $\tau(2) = \{n\}$. Since the dual of any cell generates $H^{n-2}(\mathbb{RP}^{n-2}; \mathbb{Z}_2)$, we have $w_1^{n-2}(\mathtt{Hom}\,(K_2, K_n)) = [\{\mathbb{Z}_2\tau\}]$. By functoriality of w_1 we get $w_1^{n-2}(X_{r,n}) = [\varphi(\{\mathbb{Z}_2\tau\})]$. Comparing this to our notations we derive $w_1^{n-2}(X_{r,n}) = [q(A_r)]$.

2. Proof of the Babson-Kozlov Conjecture

Lemma 2.1. *We have $\partial B_v = A_{v-1} \oplus A_{v+1}$, for any $v \in C_{2r+1}$.*

Proof. The cells in ∂B_v are obtained by taking a cell $\sigma \in B_v$ and adding x to $\sigma(w)$, for some $x \in [n]$, $w \in C_{2r+1}$. When $w \neq v \pm 1$, we get a cell τ which appears in ∂B_v twice: in $\partial\sigma_1$ and in $\partial\sigma_2$, where σ_1, σ_2 are obtained from τ by deleting one of the elements from $\tau(w)$. When $w = v \pm 1$, we also get a cell τ which appears in ∂B_v twice: in $\partial\sigma_1$ and in $\partial\sigma_2$, where σ_1, σ_2 are obtained from τ by deleting $\{x\} = \tau(v-1) \cap \tau(v+1)$ either from $\tau(v-1)$ or from $\tau(v+1)$; unless $|\tau(v-1)| = 1$ or $|\tau(v+1)| = 1$. The latter cells appear once and yield $A_{v-1} \oplus A_{v+1}$. □

Proof of Theorem 1.2. Set $K := \oplus_{i=1}^t q(B_{v_i})$, then $\partial K = \oplus_{i=1}^t \partial q(B_{v_i}) = \oplus_{i=1}^t q(\partial B_{v_i}) = \oplus_{i=1}^t (q(A_{v_i-1}) \oplus q(A_{v_i+1})) = q(A_r) \oplus q(A_0) = q(A_r)$, hence $w_1^{n-2}(X_{r,n}) = [q(A_r)] = [\partial K] = 0$. □

We remark that the Babson-Kozlov Conjecture implies the Lovász Conjecture, and that the latter was originally settled by Eric Babson and the author in [1, 3].

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Department of Computer Science, ETH Zürich, Switzerland
E-mail address: dkozlov@inf.ethz.ch